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LAMINAR FLOW HEAT TRANSFER FROM WEDGE-SHAPED BODIES WITH LIMITED HEAT CONDUCTIVITY

ULF OLSSON *

Department of Applied Thermo and Fluid Dynamics, Chalmers University of Technology, Gothenburg, Sweden

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Abstract— The problem of heat transfer to a finite wedge-shaped fin in a laminar flow is considered. The energy equations for the fluid and the solid body are solved simultaneously under the conditions of continuity in heat flux and temperature at the interface. The influence of heat conduction in the wedge is investigated using an integral method of solution. This method is shown to be in good agreement with a numerical method based on the Blasius technique and also with experiments provided that radiation is taken into account. Numerical results are given for various wedge materials and fluids.

NOMENCLATURE

B, fin breadth;
L, fin length;
m, dimensionless constant;
n, dimensionless constant;
Nu, Nusselt number;
Pr, Prandtl number;
Q, total heat flow from the fin;
Re, Reynolds number;
r, radial coordinate, Fig. 1;
T, temperature;
U, potential velocity;
u, v, boundary layer velocities;
x, coordinate along the wall, Fig. 1;
y, coordinate normal to the wall, Fig. 1;
z, transformed coordinate.

Greek letters
α, half of leading edge angle, Fig. 1;
β, dimensionless parameter;
δ, boundary layer thickness;
Δ, dimensionless parameter;
ε, emissivity;
η, dimensionless coordinate;
θ, temperature excess $= T - T_\infty$;
κ, heat diffusivity;
λ, heat conductivity;
ν, kinematic viscosity;
σ, Stefan–Boltzmann constant;
ϕ, angular coordinate, Fig. 1;
ψ, stream function.

Subscripts
b, base of the wedge;
c, convection;
f, fluid;
r, radiation;
s, solid or surface;
t, thermal boundary layer;
v, velocity boundary layer;
∞, free stream.

INTRODUCTION

IN THE analysis of heat convection to solid bodies it is common practice to prescribe the temperature, the heat flux or a combination of the two at the solid–fluid interfaces. In most real cases, however, the boundary conditions cannot be known *a priori*. It is then necessary to solve the energy equations for the fluid and the solid body

* Presently at Volvo Flygmotor AB, Trollhattan, Sweden.

simultaneously under the conditions of continuity in heat flux and temperature at the boundaries. This problem is rather complicated and it has received relatively little attention in the literature.

In some works, simplifying assumptions as regards the flow field are introduced. Thus, Perelman [1] studied the problem of slip flow around a body with distributed heat sources. Chu and Bankoff [2] treated heat transfer to slug flow between flat plates and in a circular tube. Sell and Hudson [3] considered the problem of heat transfer to a slug flow past a flat plate and Davies *et al.* [4] studied the effect of the conduction in the wall on the heat transfer to Poiseuille–Couette flow between parallel plates. Lastly, an approximate method suitable for cases in which the boundary conditions at the wall is either an approximately constant temperature or an approximately constant heat flux were given by Rotem [5].

Especially simple solutions are obtained if a uniform temperature is prescribed at the solid–fluid interfaces. Here, we will investigate the exactness of such solutions for a wedge [6] by comparing them with the solutions to the more realistic case of continuity in heat flux and temperature at the interfaces. For this investigation, we develop an integral method which is shown to be in good agreement with the numerical method based on an extension of the Blasius technique [7–9] and also with experiments provided that radiation is taken into account.

AN APPROXIMATE SOLUTION

Consider laminar flow of an incompressible fluid past a finite wedge as shown in Fig. 1. With the potential wedge flow velocity given by

$$U = u_0 x^m \tag{1}$$

where

$$m = \frac{\alpha}{\pi - \alpha} \tag{2}$$

and u_0 a constant, and with

$$\delta_t = \Delta\delta_v \tag{3}$$

where Δ is a constant, we obtain the following relation by applying the conventional momentum and energy integral methods [6]

$$\Delta\left(F(\Delta) + \frac{1260\,m}{37 + 300\,m}\,G(\Delta)\right) = \frac{37 + 300\,m}{630\,Pr}\,\frac{\int_0^L \theta_s(x)x^{(m-1)/2}\,\mathrm{d}x}{\theta_s(L)L^{(m+1)/2}} \tag{4}$$

from which Δ may be found once θ_s is known.

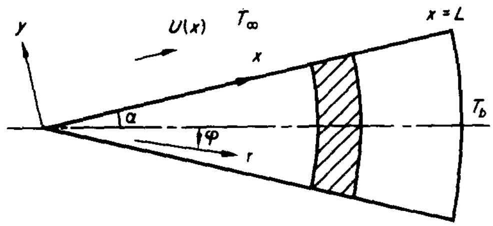

FIG. 1. The system considered.

For $\Delta < 1$ we have

$$F(\Delta) = \frac{2}{15}\Delta^2 - \frac{3}{140}\Delta^4 + \frac{1}{180}\Delta^5 \tag{5}$$

$$G(\Delta) = \frac{1}{90}\Delta^2 - \frac{1}{84}\Delta^3 + \frac{3}{560}\Delta^4 - \frac{1}{1080}\Delta^5$$

and for $\Delta > 1$

$$F(\Delta) = -\frac{3}{10} + \frac{3}{10}\Delta + \frac{2}{15}\Delta^{-1} - \frac{3}{140}\Delta^{-3} + \frac{1}{180}\Delta^{-4} \tag{6}$$

$$G(\Delta) = \frac{1}{120} - \frac{1}{180}\Delta^{-1} + \frac{1}{840}\Delta^{-3} - \frac{1}{3024}\Delta^{-4}.$$

If the wedge is sufficiently thin i.e. $m \ll 1$, then its temperature may be assumed to vary with r only, i.e. to be equal to $\theta_s(r)$. This satisfies the demand for continuity in temperature across the fluid–solid interface where $r = x$.

Then, with a linearized temperature dependence of the radiative heat flux, a heat balance for an element of the wedge as shown in Fig. 1 gives [10]

$$\lambda_s \alpha \frac{\mathrm{d}}{\mathrm{d}x}\left(x \frac{\mathrm{d}\theta_s}{\mathrm{d}x}\right) = \left[\frac{2\lambda_f}{\Delta}\left(\frac{37 + 300\, m}{1260}\frac{u_0}{\nu_f}\right)^{\frac{1}{2}} x^{(m-1)/2} + 4\varepsilon\sigma T_\infty{}^3\right]\theta_s. \tag{7}$$

If the temperature is prescribed at the base of the wedge, the boundary conditions to equation (7) will be

$$\theta_s(0) \text{ finite} \tag{8}$$

$$\theta_s(L) = \theta_b = \text{constant}. \tag{9}$$

We now have two relations, i.e. equations (4) and (7) from which Δ and $\theta_s(x)$ may be solved.

Neglecting convection, equation (7) is a Bessel differential equation which under the conditions (8) and (9) has the solution

$$\theta_s = \theta_b \frac{I_0[\beta_r(x/L)^{\frac{1}{2}}]}{I_0(\beta_r)} \tag{10}$$

where

$$\beta_r = \left(\frac{16\varepsilon\sigma L T_\infty^3}{\alpha\lambda_s}\right)^{\frac{1}{2}}. \tag{11}$$

Similarly, if radiation is neglected

$$\theta_s = \theta_b \frac{I_0[\beta_c(x/L)^{(m+1)/4}]}{I_0(\beta_c)} \tag{12}$$

where

$$\beta_c = \frac{1}{m+1}\left(\frac{32\lambda_f}{\alpha\lambda_s\Delta}\right)^{\frac{1}{2}}\left(\frac{37 + 300\, m}{1260}\frac{u_0}{\nu_f}\right)^{\frac{1}{4}} \times L^{(m+1)/4}. \tag{13}$$

In combined convection and radiation we make the transformation

$$z = x^{(m+1)/2} \tag{14}$$

and obtain the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}\theta_s}{\mathrm{d}z}\right) = \left(\frac{\beta_c^2}{4} L^{-(m+1)/2} + \frac{\beta_r^2}{L(m+1)^2} z^{(1-m)/(1+m)}\right)\theta_s. \tag{15}$$

Neglecting m with respect to 1 in the radiative terms, which are approximate in any case, makes it possible to obtain the solution in the series form

$$\theta_s = \theta_b\left[\sum_{n=0}^{\infty} C_n\left(\frac{x}{L}\right)^{n(m+1)/2}\right] \Big/ \left(\sum_{n=0}^{\infty} C_n\right) \tag{16}$$

where the coefficients C_n are given by the recursive formula

$$C_{n+2} = \frac{\beta_c^2 C_{n+1} + 4\beta_r^2 C_n}{4(n+2)^2}\quad n = 0, 1, 2\ldots \tag{17}$$

with

$$C_0 = 1 \quad C_1 = \frac{\beta_c^2}{4}. \tag{18}$$

Introducing the solutions for $\theta_s(\Delta, x)$ into equation (4) makes it possible to calculate Δ. Then, the heat flows and temperature fields may easily be found with conventional methods [6].

A NUMERICAL SOLUTION

The laminar boundary layer equations for the velocity and temperature fields in two-dimensional, stationary and incompressible flow are with viscous heating neglected

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{19}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \nu_f\frac{\partial^2 u}{\partial y^2} \tag{20}$$

$$u\frac{\partial T}{\partial x} + v\cdot\frac{\partial T}{\partial y} = \kappa_f\frac{\partial^2 T}{\partial y^2}. \tag{21}$$

They will be solved here [7–9] for the following boundary conditions

$$y = 0: u = v = 0 \tag{22}$$

$$T = T_s(x) = T_\infty + \sum_{i=0}^{\infty} t_i x^{in} \tag{23}$$

$$y = \infty : u = U(x) = x^m \sum_{j=0}^{\infty} u_j x^{jn} \tag{24}$$

$$T = T_\infty \tag{25}$$

where m, n, u_i and t_i are arbitrary numbers and where u_0 and t_0 are nonvanishing.

Now introducing the stream function ψ as

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \tag{26}$$

the equation of continuity (19) is identically satisfied. Then the stream function is expanded in series form as

$$\psi = x^{(m+1)/2}\, u_0^{\frac{1}{2}}\, v_f^{\frac{1}{2}} \sum_{j=0}^{\infty} \left(\frac{u_1}{u_0}\right)^j f_j(\eta)\, x^{jn} \tag{27}$$

where

$$\eta = \tfrac{1}{2} y x^{(m-1)/2}\, u_0^{\frac{1}{2}}\, v_f^{-\frac{1}{2}}. \tag{28}$$

Inserting U and ψ in equations (20), (22) and (24) and singling out coefficients in x gives a system of recursive ordinary differential equations and boundary conditions for the functions $f_j(\eta)$. Having solved this system, the velocity boundary layer is known.

Because equation (21) is linear, solutions for each term in equation (23) may be found separately and then superposed to form the total solution. It is found suitable to introduce the solution for one such term in the form

$$\theta_i = t_i \sum_{j=0}^{\infty} \left(\frac{u_1}{u_0}\right)^{j-i} F_{ij}(\eta) x^{jn};\; F_{ij} = 0\; i > j. \tag{29}$$

Introducing equations (27) and (29) into equations (21), (23) and (25) and singling out coefficients for x gives a recursive system of ordinary differential equations and boundary conditions for the functions $F_{ij}(\eta)$. Thus, these functions may be found for each term in equation (23) and a summation of the solutions equation (29) gives the total solution. The detailed calculations are rather laborious and will be excluded here.

Now turning to the wedge-shaped solid body the stationary heat conduction equation will be, in cylindrical coordinates

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2} = 0. \tag{30}$$

With continuity in temperature across the solid–fluid interface, the boundary conditions are

$$r = L : \theta = \theta_b = \sum_{i=0}^{\infty} t_i L^{in} \tag{31}$$

$$\phi = \alpha : \theta = \theta_s(x) = \sum_{i=0}^{\infty} t_i x^{in}. \tag{32}$$

Furthermore the solution should be finite at $r = 0$ and symmetric with respect to $\phi = 0$. Note that equation (31) restricts the choice of the coefficients t_i.

A solution which satisfies these conditions may be written

$$\theta = \sum_{i=0}^{\infty} A_i \left(\frac{r}{L}\right)^{a_i} \cos a_i\phi + \sum_{i=0}^{\infty} B_i \left(\frac{r}{L}\right) \cos b_i\phi \tag{33}$$

where

$$a_i = in \tag{34}$$

$$b_i = (2i+1)\frac{\pi}{2\alpha} \tag{35}$$

$$A_i = \frac{t_i L^{in}}{\cos in\,\alpha} \tag{36}$$

$$B_i = (-1)^i \frac{4}{\pi}\frac{1}{2i+1} \sum_{j=1}^{\infty} \frac{j^2 n^2}{j^2 n^2 - b_i^2}\, t_j L^{jn}. \tag{37}$$

Here the coefficients B_i have been obtained through developing both membra of the boundary condition (31) into a series of $\cos b_i\phi$.

We have now expressed the temperatures in the fluid and in the solid into the as yet unknown coefficients t_i. These coefficients may be obtained through satisfying the demand of continuity in the heat flux across the fluid–solid interface. This is achieved here by selecting the coefficients so that the heat fluxes in the fluid and in the solid as obtained from equations (29) and (33) respectively match each other as well as possible according to the discrete least square method.

A COMPARISON BETWEEN SOLUTIONS

The numerical and approximate methods described previously were applied to the problem of heat transfer from wedge-shaped fins whose surface temperature distributions had been determined experimentally [11].

Black-painted fins of copper and iron with the following data were considered

$\alpha = 0{\cdot}041$ rad, $L = 0{\cdot}244$ m, $\varepsilon = 0{\cdot}98$

Copper: $\lambda_s = 405$ W/m°C, $U(L) = 3{\cdot}05$ m/s, $T_\infty = 300{\cdot}5$°K

Iron: $\lambda_s = 65$ W/m°C, $U(L) = 3{\cdot}13$ m/s, $T_\infty = 300$°K.

It was found that the radiative heat flux could not be neglected. In the numerical method a value of $n = 1$ was used together with terms up to and including t_5 and B_{100} in the series (29) and (33) respectively. Equation (1) was used for the potential velocity U.

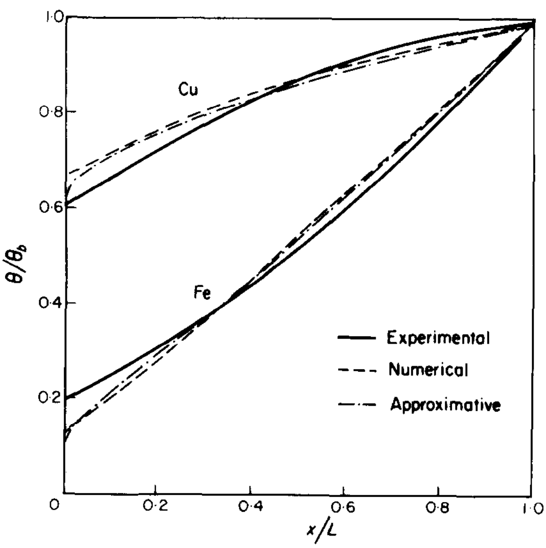

FIG. 2. Comparison of solutions with air as cooling medium.

Figure 2 compares the results from the approximate, numerical and experimental methods. As is seen, both the approximate and the numerical method gives a relatively exact description of the measured temperature distributions. The difference between the two methods is less than the difference between either of them and the experimental results. We therefore consider the approximate method to be sufficiently accurate for our purpose.

RESULTS AND DISCUSSION

In the previous comparisons with experiments at $\varepsilon = 0{\cdot}98$ the radiating effects could not be neglected. Usually, however, the emissivities are lower and radiation has less importance. Therefore we shall study only convective heat transfer here.

As is seen from equation (12), the temperature distribution in the wedge is governed by the parameter β_c which may be rewritten as.

$$\beta_c = \frac{1}{m+1}\left(\frac{32\lambda_f L}{\alpha\lambda_s\delta_t(L)}\right)^{\frac{1}{2}}. \tag{38}$$

Thus, β_c may be understood as a ratio of the axial resistance for heat flow in the wedge to the transverse resistance in the flow. Since the heat will flow most easily along the path of least resistance, it follows that a lower value of β_c should give a more uniform temperature distribution. This may also be seen from equation (12). Therefore, the case of a uniform temperature distribution which is usually adopted in heat transfer calculations is realizable only at e.g. infinite heat conductivity in the wall. The influence of a finite heat conductivity is then to give rise to axial temperature variations. This tendency has already been found in Fig. 2.

We may easily derive the following characteristic dimensionless number

$$Nu_x Re_x^{-\frac{1}{2}} = \frac{1}{\Delta}\left(\frac{37 + 300\,m}{315}\right)^{\frac{1}{2}}. \tag{39}$$

In practical calculations, it is usually assumed that the heat flow into the fluid may be found without taking into account the heat conduction in the wedge. The temperature is then prescribed at the surface and the corresponding value of $Nu_x Re_x^{-\frac{1}{2}}$ is found. Usually $Nu_x Re_x^{-\frac{1}{2}}$ is taken with its value at a uniform surface temperature [6], after what Δ may be found from equation (39) and the surface temperature θ_s from equation (12).

But such a choice is clearly unrealistic because, as was said previously, the surface temperature will not be uniform at finite heat conductivities.

In the coupled theory, i.e. taking into account the heat conduction in the wedge, Δ is calculated from equations (4) and (12). The corresponding values of $Nu_x Re_x^{-\frac{1}{2}}$, as calculated from equation (39), are shown in Fig. 3 for some possible

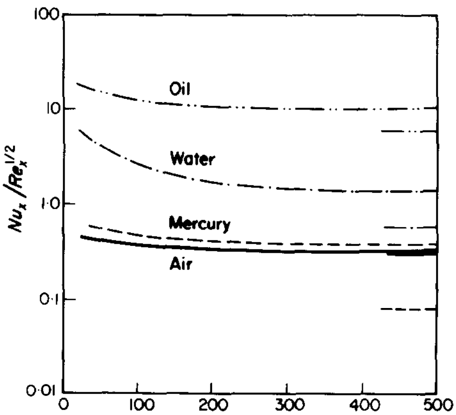

FIG. 3. The variation of $Nu_x Re_x^{-\frac{1}{2}}$ with the heat conductivity of the wedge for various cooling media.

cooling media. The data used in this and the following calculations are, excluding λ_s, the same as those previously given for iron. As is seen, $Nu_x Re_x^{-\frac{1}{2}}$ varies considerably with λ_s, the values of [6] being obtained asymptotically at infinite heat conductivities.

These variations indicate that using the asymptotic values could lead to large errors in calculations of the temperature distributions and heat flows. With air as the cooling medium, Fig. 4 compares the temperature distributions obtained by using the asymptotic values of $Nu_x Re_x^{-\frac{1}{2}}$, i.e. those given in [6], with the corresponding results from the coupled theory. It is found that the differences may be considerable at lower heat conductivities. Those differences should be still larger for the other cooling media because air shows the least variations of $Nu_x Re_x^{-\frac{1}{2}}$ in Fig. 3. Of more direct interest is the

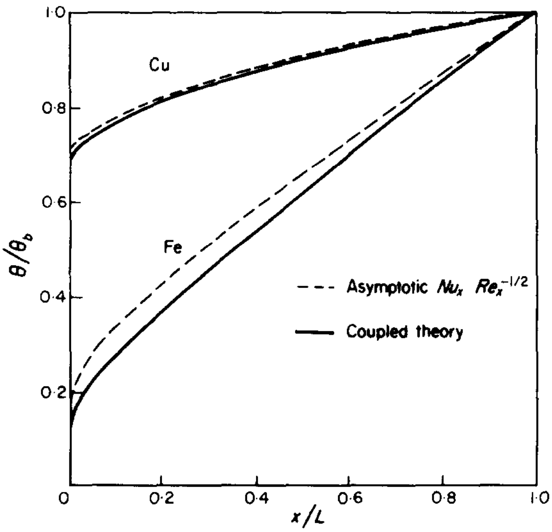

FIG. 4. Comparison of temperature distributions obtained by using the asymptotic value of $Nu_x Re_x^{-\frac{1}{2}}$ with corresponding results from coupled theory. Air as cooling medium.

total heat flow from the wedge. As is seen in Fig. 5 it is underestimated if asymptotic values of $Nu_x Re_x^{-\frac{1}{2}}$ are used. For air, this approximation gives relatively exact results especially at higher heat conductivities. However, it leads to large errors for the other cooling media.

Studies of the parameter β_c indicates that it is an increasing function of $\lambda_f/\alpha\lambda_s$, Re_L and Pr. Recalling the definition of β_c, it follows that the temperature distribution in the wedge should be more uniform at larger wedge angles, shorter fin lengths and lower fluid velocities. Also, since $Nu_x Re_x^{-\frac{1}{2}}$ defines the heat flux from the surface, it seems plausible that it should increase with β_c i.e. with $\lambda_f/\alpha\lambda_s$. Therefore, it should vary more with λ_s at larger values of λ_f. On the other hand, its asymptotic value is determined primarily by Pr, [6]. These tendencies are found in Fig. 3. Mercury has e.g. a much higher heat conductivity λ_f than has air while its Prandtl number Pr is considerably lower. Furthermore, the errors made when neglecting the heat conduction in the wedge should be smaller in systems with less variations of $Nu_x Re_x^{-\frac{1}{2}}$ with λ_s. Therefore, smaller values of $\lambda_f/\alpha\lambda_s$ and Re_L should give smaller errors.

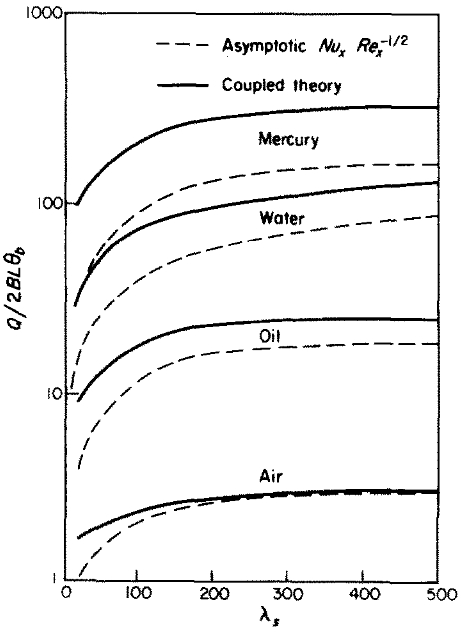

FIG. 5. The total heat flow from the wedge using asymptotic values of $Nu_x Re_x^{-\frac{1}{2}}$ and coupled theory respectively.

In conclusion, it follows from what has been said here that it is necessary to solve the coupled heat transfer problem in order to get a good description of the temperature distributions and heat flows in a body in a laminar flow. The assumption of a uniform surface temperature leads to underestimations of the cooling capacity of the fluid. Thus, this should be a conservative method. Lastly, for wedges, the approximate method given here should be a valuable tool of analysis because it is almost exact and demands relatively little calculation.

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TRANSFERT THERMIQUE A UN ECOULEMENT LAMINAIRE PAR DES CORPS EN FORME DE DIEDRE AVEC CONDUCTIVITE THERMIQUE LIMITEE

Résumé—On considère le problème du transfert thermique d'une ailette en forme de dièdre dans un écoulement laminaire. Les équations d'énergie pour le fluide et le corps solide sont résolues simultanément sous les conditions de continuité du flux thermique et de la température à l'interface.

On a analysé l'influence de la conduction thermique dans le dièdre à l'aide d'une méthode intégrale de résolution. On montre que cette méthode est en bon accord avec une méthode numérique basée sur la technique de Blasius et aussi avec des expériences dans lesquelles intervient le rayonnement.

Des résultats numériques sont donnés pour divers matériaux du dièdre et divers fluides.

WÄRMEÜBERTRAGUNG IN LAMINARER STRÖMUNG AN KEILFÖRMIGEN RIPPEN MIT ENDLICHER WÄRMELEITFÄHIGKEIT

Zusammenfassung—Es wird das Problem des Wärmeübergangs auf eine endliche keilförmige Rippe in laminarer Strömung untersucht. Die Energiegleichungen für Fluid und Rippe werden gleichzeitig gelöst unter den Bedingungen eines kontinuierlichen Verlaufs des Wärmestroms und der Temperatur an der Trennfläche. Der Einfluss der Wärmeleitung in der Rippe wird untersucht mit Hilfe einer integralen Lösungsmethode. Es zeigt sich, dass diese Methode gut übereinstimmt mit einer numerischen Methode nach Blasius und auch mit Experimenten ohne Berücksichtigung der Strahlung. Die Ergebnisse in Zahlen werden für verschiedene Rippenmaterialien und verschiedene Strömungsmedien aufgeführt.

ТЕПЛООТДАЧА ТЕЛ В ФОРМЕ КЛИНА ПРИ ЛАМИНАРНОМ ОБТЕКАНИИ С УЧЕТОМ ТЕПЛОПРОВОДНОСТИ

Аннотация—Рассмотрена задача переноса тепла к ребру клинообразной формы конечной длиныу равнения энергии для жидости и твердого тела решаются совместно при условии непрерывности теплового потока и температуры на поверхности раздела. Влияние теплопроводности в клине исследуется с помощью интегрального метода. Показано, что этот метод дает хорошее соответствие с численным решением метода Блазиуса, а также с экспериментальными данными с учетом излучения. Приведены численные результаты для различных материалов клина и различных жидкостей.